Practice Final

1 Notes on this exam

The final, like this practice final, will be 10 questions long (some questions may have multiple parts).

The real final will be similar in many ways to this one, though some topics will be chosen differently.

2 Problems

1. Identify whether each of the following statements is true or false; please write "T" or "F" for your answer. No justification is needed.

   (a) If A is $n \times n$, then $\dim(\operatorname{col} A) + \dim(\operatorname{nul} A) = n$.

   (b) Every $n \times n$ matrix A is diagonalizable.

   (c) A regular stochastic matrix never has any 0 entries.

   (d) An orthogonal matrix U always satisfies $U^T = U$.

   (e) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.

2. Let
$$A = \begin{bmatrix} 1 & h \\ 3 & 12 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
   For what values of h will the system $A\mathbf{x} = \mathbf{b}$ have no solution?

3. (a) Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which acts as
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ 4x_2 \end{bmatrix}.$$
   Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

(b) Consider the quadratic form $Q : \mathbb{R}^2 \to \mathbb{R}$ which acts as

$$Q\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = 2x_1^2 + 6x_1x_2 + 2x_2^2 \ .$$

Find a symmetric matrix B such that $Q(\mathbf{x}) = \mathbf{x}^T B\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^2$.

4. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} .$$

Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$.

5. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

Show that $\det(A + B) = \det(A) + \det(B)$ if and only if $a + d = 0$.

6. Let $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} .$$

Use the Gram-Schmidt procedure to find an orthonormal basis for W.

7. Suppose we have a Markov chain $P\mathbf{x}_i = \mathbf{x}_{i+1}$, where

$$P = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix} .$$

(a) If $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what are $\mathbf{x}_1$ and $\mathbf{x}_2$?

(b) What is the long-run behavior of the Markov chain? That is, find a vector $\mathbf{q}$ such that $\mathbf{x}_k \to \mathbf{q}$ as $k \to \infty$.

8. Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^n$. The matrix $A = I_n + \mathbf{v}\mathbf{v}^T$ is called a "rank-one perturbation of the identity", and a well-known formula states that A is invertible, and its inverse satisfies the formula

$$A^{-1} = I_n + \alpha\mathbf{v}\mathbf{v}^T$$

for some real number α. Find α.

9. Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 & -5 \\ 0 & 0 & -1 & 2 & 2 \\ 2 & 6 & -5 & 4 & -8 \end{bmatrix}.$$

   (a) Find a basis for $\operatorname{col} A$.

   (b) Find a basis for $\operatorname{nul} A$.

   (c) Find a basis for $\operatorname{row} A$.

   (d) Verify that $\dim \operatorname{col} A$ and $\dim \operatorname{row} A$ have the relationship you expect.

10. Consider the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

Orthogonally diagonalize A. That is, find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{-1}$.